A FEW WEIGHT SYSTEMS ARISING FROM INTERSECTION GRAPHS

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ABSTRACT

We show that the adjacency matrices of the intersection graphs of chord diagrams satisfy the 2-term relations of Bar-Natan and Garoufalides [2], and hence give rise to weight systems. Among these weight systems are those associated with the Conway and HOMFLYPT polynomials. We extend these ideas to looking at a space of *marked* chord diagrams modulo an extended set of 2-term relations, define a set of generators for this space, and again derive weight systems from the adjacency matrices of the (marked) intersection graphs. Among these weight systems are those associated with the Kauffman polynomial.

Contents

1 Introduction

Finite type invariants have received a lot of attention over the past decade. One reason for this is that they provide a common framework for many of the most powerful knot invariants, such as the Conway, Jones, HOMFLYPT and Kauffman invariants. The framework also allows us to study these invariants using elementary combinatorics, by looking at associated functionals (called *weight systems*) on spaces of *chord diagrams*. This provides a new ways of describing the invariants.

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The modest goal of this paper is to define a few weight systems in terms of the adjacency matrix of the intersection graph of the chord diagrams, and to show that among these weight systems are those associated with the Conway, HOMFLYPT and Kauffman polynomials, in both their framed and unframed incarnations. This gives us new formulas for the weight systems associated to these important knot invariants. We build on ideas of Bar-Natan and Garoufalides [2], who first found the formula we give for the Conway polynomial.

In section 2 we will review the necessary background for the paper, including finite type invariants, the 2-term relations introduced by Bar-Natan and Garoufalides, intersection graphs of chord diagrams and Lando's graph bialgebra. In section 3 we will study the adjacency matrix of the intersection graph, and show that the weight systems associated with the Conway and HOMFLYPT polynomials can be defined in terms of the determinant and rank of this matrix. In section 4 we look at *marked* chord diagrams and define an extended set of 2-term relations on these diagrams. We give an explicit set of generators for the space of marked chord diagrams modulo these relations. Finally, we show that the weight system associated with the Kauffman polynomial can be defined in terms of the rank of the adjacency matrix of marked chord diagrams.

Remark: The result for the Conway polynomial (Theorem 4) has been previously proved by Bar-Natan and Garoufalidis [2], but is included here for completeness and to place it in the context of Lando's bialgebra. After distributing the first version of this paper [11], the author discovered that the adjacency matrix of an intersection graph has also been studied by Soboleva [15], who has also proven Theorem 5, and a weaker version of Theorem 11. The intersection graphs we study are also related to the *trip matrix* of a knot, studied by Zulli [17].

2 Preliminaries

2.1 Finite Type Invariants

In 1990, V.A. Vassiliev introduced the idea of *Vassiliev* or *finite type* knot invariants, by looking at certain groups associated with the cohomology of the space of knots. Shortly thereafter, Birman and Lin [3] gave a combinatorial description of finite type invariants. We will give a brief overview of this combinatorial theory. For more details, see Bar-Natan [1].

A *knot* is an embedding of the circle S^1 into the 3-sphere S^3. A *knot invariant* is a map from these embeddings to some set which is invariant under isotopy of the embedding. We will also consider invariants of *regular* isotopy, where the isotopy preserves the *framing* of the knot (i.e. a chosen section of the normal bundle of the knot in S^3). We first note that we can extend any knot invariant to an invariant of *singular* knots, where a singular knot is an immersion of S^1 in 3-space which is an embedding except for a finite number of isolated double points. Given a knot invariant v, we extend it via the relation:

An invariant v of singular knots is then said to be of *finite type*, specifically of *type n*, if v is zero on any knot with more than n double points (where n is a finite nonnegative integer). The smallest such n is called the *order* of v. We denote by V_n the vector space over $\mathbb{C}$ generated by (framing-independent) finite type invariants of type n (i.e., whose order is $\leq n$). We can completely understand the space of finite type invariants by understanding all of the vector spaces V_n/V_{n-1}. An element of this vector space is completely determined by its behavior on knots with exactly n singular points. In addition, since such an element is zero on knots with more than n singular points, any other (non-singular) crossing of the knot can be changed without affecting the value of the invariant. This means that elements of V_n/V_{n-1} can be viewed as functionals on the space of *chord diagrams*:

Definition 1 A **chord diagram of degree n** *is an oriented circle, together with n chords of the circles, such that all of the 2n endpoints of the chords are distinct. The circle represents a knot, the endpoints of a chord represent 2 points identified by the immersion of this knot into 3-space. The diagram is determined by the order of the 2n endpoints.*

(1-term relation)

(4-term relation)

Figure 1: The 1-term and 4-term relations. No other chords have endpoints on the arcs labeled with a *. In the 4-term relations, all other chords of the four diagrams are the same.

Functionals on the space of chord diagrams which are derived from finite type knot invariants will satisfy certain relations. This leads us to the definition of a *weight system*:

Definition 2 *A* **weight system of degree n** *is a linear functional W on the space of chord diagrams of degree n (with values in an associative commutative ring $\mathbf{K}$ with unity) which satisfies the 1-term and 4-term relations, shown in Figure 1.*

It can be shown (see [3, 1, 16]) that the space of weight systems of degree n is isomorphic to V_n/V_{n-1}. For convenience, we take the dual approach, and simply study the space of chord diagrams of degree n modulo the 1-term and 4-term relations. The 1-term relation is occasionally referred to as the "framing-independence" relation, because it arises from the framing-independence of the invariants in V_n (essentially, from the first Reidemeister move). Since most of the interesting structure of the vector spaces arises from the 4-term relation, it is common to look at the more general setting of invariants of regular isotopy, and consider the vector space A_n of chord diagrams of degree n modulo the 4-term relation alone. We will call the space W_n of linear functionals on A_n the space of *regular weight systems* of degree n. We will let $\widehat{A_n}$ denote the vector space of chord diagrams modulo both the 1-term and 4-term relations, and $\widehat{W_n}$ denote the space of functionals on $\widehat{A_n}$, the unframed weight systems.

It is useful to combine all of these spaces into a graded module $A = \bigoplus_{n \geq 1} A_n$ via direct sum. We can give this module a bialgebra (or Hopf algebra) structure by defining an appropriate product and co-product:

- We define the product $D_1 \cdot D_2$ of two chord diagrams D_1 and D_2 as their connect sum. This is well-defined modulo the 4-term relation (see [1]).

- We define the co-product $\Delta(D)$ of a chord diagram D as follows:

$$\Delta(D) = \sum_J D'_J \otimes D''_J$$

where J is a subset of the set of chords of D, D'_J is D with all the chords in J removed, and D''_J is D with all the chords *not* in J removed.

It is easy to check the compatibility condition $\Delta(D_1 \cdot D_2) = \Delta(D_1) \cdot \Delta(D_2)$.
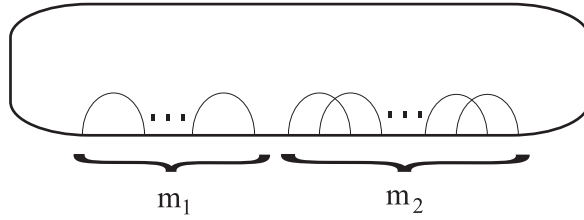
Figure 2: The 2-term relations

Figure 3: Example of an (m_1, m_2)-caravan

There is a natural *deframing* map $\phi : A \otimes A \to A$, defined by:

$$\phi(D_1 \otimes D_2) = (-\Theta)^{deg(D_1)} \cdot D_2$$

Here Θ represents the chord diagram consisting of a single chord. This map gives a canonical projection $\widehat{\ } : W_n \to \widehat{W_n}$, defined by $\widehat{W}(D) = W(\phi(\Delta(D)))$ (see [1], Exercise 3.16).

2.2 2-Term relations

Of course, any particular weight system will satisfy relations in addition to the 1-term and 4-term relations, and it can be useful to look at weight systems which lie in the subspaces determined by these additional relations. In particular, Bar-Natan and Garoufalides [2] noted that the weight system associated with the Conway polynomial satisfies the *2-term* relations in Figure 2. Clearly, these relations imply that the weight system satisfies the 4-term relation as well. As a result, the product and coproduct of section 2.1 are still well-defined. So we can give the vector space of chord diagrams modulo the 2-term relations the structure of a bialgebra. We will denote this bialgebra (and the underlying vector space) by B. There is a natural projection from A to B.

Bar-Natan and Garoufalides also showed that B is generated (as a vector space) by *(m_1, m_2)-caravans* of m_1 "one-humped camels" (isolated chords which intersect no other chords) and m_2 "two-humped camels" (pairs of chords which intersect each other, but no other chords). An example of such a caravan is shown in Figure 3.

2.3 Intersection Graphs

Definition 3 *Given a chord diagram D, we define its **intersection graph** $\Gamma(D)$ as the graph such that:*

- $\Gamma(D)$ *has a vertex for each chord of D.*

- *Two vertices of $\Gamma(D)$ are connected by an edge if and only if the corresponding chords in D intersect, i.e. their endpoints on the bounding circle alternate.*

For example:
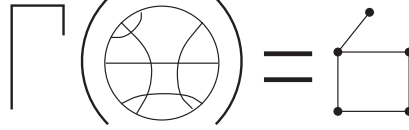
Note that these graphs are simple (i.e. every edge has two distinct endpoints, and there is at most one edge connecting any two vertices). These graphs are also known as *circle graphs*, and have been studied extensively by graph theorists. A combinatorial classification of circle graphs has been given by Bouchet [4].

A circle graph can be the intersection graph for more than one chord diagram. For example, there are three different chord diagrams with the following intersection graph:
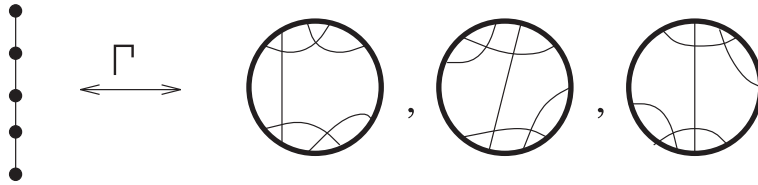
However, these chord diagrams are all equivalent modulo the 4-term relation. Chmutov, Duzhin and Lando [6] conjectured that intersection graphs actually determine the chord diagram, up to the 4-term relation. In other words, they proposed:

Conjecture 1 *If D_1 and D_2 are two chord diagrams with the same intersection graph, i.e. $\Gamma(D_1) = \Gamma(D_2)$, then for any weight system W, $W(D_1) = W(D_2)$.*

This Intersection Graph Conjecture is now known to be false in general. Morton and Cromwell [13] found a finite type invariant of type 11 which can distinguish some mutant knots, and Le [8] and Chmutov and Duzhin [5] have shown that mutant knots cannot be distinguished by intersection graphs. However, the conjecture is true in many special cases, and the exact extent to which it fails is still unknown, and potentially very interesting.

The conjecture is known to hold in the following cases [6]:

- For chord diagrams with 8 or fewer chords (checked via computer calculations);

- For the weight systems coming from the defining representations of Lie algebras *gl(N)* or *so(N)* as constructed by Bar-Natan in [1];

- When $\Gamma(D_1) = \Gamma(D_2)$ is a tree (or, more generally, a linear combination of forests);

- When $\Gamma(D_1) = \Gamma(D_2)$ has a single loop (see [10]).

The second item above includes the weight systems arising from the Conway, HOMFLYPT and Kauffman polynomials. A main goal of this paper is to find explicit formulas for these weight systems in terms of intersection graphs.

2.4 Lando's graph bialgebra

Lando [7] has given more structure to the questions surrounding intersection graphs by extending the map Γ to a homomorphism between the bialgebra A of chord diagrams and a particular bialgebra of graphs. Lando's bialgebra of graphs is constructed by defining an analogue of the 4-term relation for graphs, as follows:

Definition 4 *[7] Consider the graded vector space (over $\mathbb{C}$) of formal linear combinations of graphs, graded by the number of vertices in the graphs. For any graph G and vertices A and B in $V(G)$ we impose on the vector space the relation:*

$$G - G'_{AB} - \widetilde{G}_{AB} + \widetilde{G}'_{AB} = 0$$

where G'_{AB} is the result of complementing the edge AB in G (i.e. adding or removing it), $\widetilde{G}_{AB}$ is the result of complementing the edge AC for every vertex C in $V(G)$ which is adjacent to B and $\widetilde{G}'_{AB}$ is the result of complementing the edge AB in $\widetilde{G}_{AB}$. Here is an example of such a relation:

The bialgebra F is defined as this graded vector space, together with a product and a coproduct. The product is simply disjoint union of graphs. The coproduct is a map $\mu : F \to F \otimes F$, defined as follows. For any graph G, and subset $J \subseteq V(G)$ of its vertices, let G_J denote the subgraph induced by J. Then:

$$\mu(G) = \sum_{J \subseteq V(G)} G_J \otimes G_{V(G) \setminus J}$$

An example is shown below:

It is now easy to show that Γ extends to a bialgebra homomorphism from A to F (see [7]).

We can easily extend Lando's results to include the 1-term relation and framing-independent invariants. We define the algebra $\widehat{F}$ to be simply F modulo graphs with isolated vertices (these correspond to the isolated chords of the 1-term relation for chord diagrams). It is then trivial to show that Γ extends to a bialgebra homomorphism from $\widehat{A}$ to $\widehat{F}$.

A *regular graph weight system* is a linear functional $\gamma : F \to \mathbb{C}$ (Lando called these functionals *4-invariants*) . Then, given any regular graph weight system γ, $\gamma \circ \Gamma : A \to \mathbb{C}$ is a regular weight system. Similarly, if we define a *graph weight system* to be a linear functional of $\widehat{F}$, then for any graph weight system α, $\alpha \circ \Gamma$ will be a weight system.

Just as for chord diagrams, there is a natural *deframing* map $\phi : F \otimes F \to F$, defined by:

$$\phi(G_1 \otimes G_2) = (-\bullet)^{deg(G_1)} \cdot G_2$$

Here $\bullet$ represents the trivial graph consisting of a single vertex and no edges. This map gives a canonical projection$\widehat{}$: $F^* \to \widehat{F}^*$, defined by $\widehat{\gamma}(G) = \gamma(\phi(\mu(G)))$.

3 The Adjacency Matrix of an Intersection Graph

In this section we will show that the determinant and rank of the adjacency matrix of a graph (over $\mathbf{Z}_2$) are regular graph weight systems, and that the determinant is, in addition, a graph weight system. We will do this by showing that the isomorphism class of the adjacency matrix (as a symmetric bilinear form over $\mathbf{Z}_2$) satisfies 2-term relations analagous to those in section 2.2. We will then show that these weight systems are essentially the same as those associated with the Conway and HOMFLYPT polynomials.

3.1 Graph Weight Systems from the Adjacency Matrix

We begin by recalling the definition of the adjacency matrix of a graph.

Definition 5 *Given a graph G with n vertices, labeled $\{v_1, ..., v_n\}$, the* **adjacency matrix of G***, or adj(G), is the symmetric $n \times n$ matrix defined by:*

$$adj(G)_{ij} = \begin{cases} 1 \; if \; v_i \; and \; v_j \; are \; connected \; by \; an \; edge \; in \; G \\ \qquad\qquad 0 \; otherwise \end{cases}$$

In the case of a simple graph, the diagonal entries of the matrix will all be 0.

This matrix can be viewed as a symmetric bilinear form over $\mathbf{Z}_2$. If we permute the labels on the vertices of G, we change the matrix $adj(G)$ by the corresponding permutations of the rows and columns. But this does not change the isomorphism class of the form (see [14]). So, as an isomorphism class of symmetric bilinear forms, the adjacency matrix of an unlabeled graph is well-defined. From Milnor and Husemoller [14], we know that the determinant and rank of the matrix are invariants of the isomorphism class of the form, and hence are well-defined invariants of the graph. This leads us to define the following functions on graphs:

Definition 6 *Given a graph G, we define the* **determinant of G** *and the* **rank of G** *as follows:*

$$det(G) = det(adj(G)) \in \mathbf{Z}_2$$

$$rank(G) = rank(adj(G))$$

We extend these functions linearly to get $\mathbf{Z}$-valued functionals on the space of graphs. We will also call these extensions the determinant and rank. We will see that the determinant gives a $\mathbf{Z}$-valued graph weight system, and the rank gives a $\mathbf{Z}$-valued regular graph weight system (the rank does not satisfy the 1-term relation). We first show that both functionals are regular graph weight systems. To do this, we will show that they satisfy *2-term* relations, analogous to those in section 2.2, defined as follows. Consider graphs $G, G'_{AB}, \widetilde{G}_{AB}, \widetilde{G}'_{AB}$ as in section 2.4. Then the 2-term relations are:

$$G - \widetilde{G}_{AB} = 0$$

$$G'_{AB} - \widetilde{G}'_{AB} = 0$$

It is clear that any functional which satisfies these 2-term relations will also satisfy the 4-term relation. So the vector space E of graphs modulo the 2-term relations can be given the structure of a bialgebra, using the same product and coproduct as for F. There is a natural projection from F to E. Moreover, the pullback by Γ of any functional on E will be a functional on B (defined in section 2.2).

Theorem 1 *The isomorphism class of the adjacency matrix of a graph satisfy the 2-term relations above.*

PROOF: Consider two vertices A and B, giving rise to the four graphs $G, G'_{AB}, \widetilde{G}_{AB}, \widetilde{G}'_{AB}$. We want to show that $adj(G) \cong adj(\widetilde{G}_{AB})$ and $adj(G'_{AB}) \cong adj(\widetilde{G}'_{AB})$. The easiest way to do this is simply to write down the matrices explicitly. The vertices of G other than A and B can be partitioned into four sets $S_{AB}, S_A, S_B,$ and S_0, where S_{AB} contains those vertices adjacent to both A and B in G, S_A contains those vertices adjacent to A but not B in G, S_B contains those vertices adjacent to B but not A in G, and S_0 contains those vertices adjacent to neither A nor B in G.

The adjacency matrices for the four graphs, with respect to the basis $\{A, B, S_{AB}, S_A, S_B, S_0\}$, are shown below. We assume that A and B are connected by an edge in G (if not, simply interchange G and G'). Here I and O represent a row or column of 1's and 0's respectively:

$$adj(G) = \begin{bmatrix} 0 & 1 & I & I & O & O \\ 1 & 0 & I & O & I & O \\ I & I & * & * & * & * \\ I & O & * & * & * & * \\ O & I & * & * & * & * \\ O & O & * & * & * & * \end{bmatrix} \cong \begin{bmatrix} 0 & 1 & O & I & I & O \\ 1 & 0 & I & O & I & O \\ O & I & * & * & * & * \\ I & O & * & * & * & * \\ I & I & * & * & * & * \\ O & O & * & * & * & * \end{bmatrix} = adj(\widetilde{G}_{AB})$$

$$\mathrm{L_+} = \qquad \mathrm{L_-} = \qquad$$
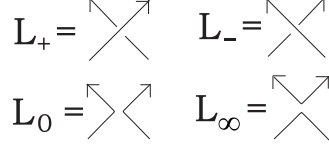
$$\mathrm{L_0} = \qquad \mathrm{L_\infty} = \qquad$$

Figure 4: Diagrams of the skein relation.

$$adj(G'_{AB}) = \begin{bmatrix} 0 & 0 & I & I & O & O \\ 0 & 0 & I & O & I & O \\ I & I & * & * & * & * \\ I & O & * & * & * & * \\ O & I & * & * & * & * \\ O & O & * & * & * & * \end{bmatrix} \cong \begin{bmatrix} 0 & 0 & O & I & I & O \\ 0 & 0 & I & O & I & O \\ O & I & * & * & * & * \\ I & O & * & * & * & * \\ I & I & * & * & * & * \\ O & O & * & * & * & * \end{bmatrix} = adj(\widetilde{G}'_{AB})$$

The isomorphisms are just the result of adding the second row (and column) of the matrix on the left to its first row (and column), modulo 2. So the isomorphism classes of the adjacency matrices of the graphs satisfy the 2-term relations. $\square$

Corollary 1 *The (linear extensions of) the rank and determinant of a graph are regular graph weight systems.*

Theorem 2 *The (linear extension of) the determinant of a graph is a graph weight system.*

PROOF: We need to show that the determinant of a graph satisfies the 1-term relation - i.e. that it is trivial on graphs with isolated vertices. Let G be a graph with an isolated vertex, v, and let $G* = G - \{v\}$. Then the adjacency matrix for G can be represented:

$$adj(G) = \begin{bmatrix} 0 & 0 \\ 0 & adj(G*) \end{bmatrix}$$

Since there is a row (and column) of 0's, $det(G) = 0$, so the determinant satisfies the 1-term relation. $\square$

As we mentioned earlier, the rank of a graph does *not* satisfy the 1-term relation, so it is not a graph weight system. However, we can use the canonical projection from section 2.4 to construct a graph weight system from the rank. In fact, we will construct a *polynomial* graph weight system by beginning with the invariant $R(G)(x) = x^{rank(G)}$, whose linear extension is clearly also a regular graph weight system. To apply our projection, it suffices to note that $rank(G_1 \cdot G_2) = rank(G_1) + rank(G_2)$, and so $rank(\bullet^{deg(G_1)} \cdot G_2) = deg(G_1)rank(\bullet) + rank(G_2) = rank(G_2)$.

Theorem 3 *Given a graph G, we define a polynomial $\widehat{R}(G)(x)$ as follows. Here J is a subset of the vertices of G, $|J|$ is the size of J, n is the total number of vertices in G, and G_J is the subgraph of G induced by J:*

$$\widehat{R}(G)(x) = \sum_J (-1)^{n-|J|} x^{rank(G_J)}$$

This polynomial is the canonical projection of $R(G)$, and so its linear extension to a $\mathbf{Z}[x]$-valued functional on the space of graphs is a graph weight system.

3.2 The Conway and HOMFLYPT weight systems

The Conway polynomial Δ of a link is a power series $\Delta(L) = \sum_{n \geq 0} a_n(L) z^n$. It can be computed via the skein relation (where L_+, L_-, L_0 are as in Figure 4):

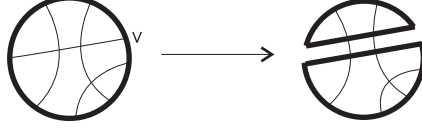$$\Delta(L_+) - \Delta(L_-) = z\Delta(L_0)$$

Figure 5: Surgery on a chord v

$$\Delta(unlink\ of\ k\ components) = \begin{cases} 1\ if\ k = 1 \\ 0\ if\ k > 1 \end{cases}$$

The coefficient a_n is a finite type invariant of type n (see [3], [1]), and therefore defines a weight system b_n of degree n. The collection of all these weight systems is called the Conway weight system, denoted C. Consider a chord diagram D, together with a chord v. Let D_v be the result of *surgery on* v, i.e. replacing v by an untwisted band, and then removing the interior of the band and the intervals where it is attached to D, as shown in Figure 5 (so D_v may have multiple boundary circles). The skein relations for the Conway polynomial give rise to the following relations for C:

$$C(D) = C(D_v)$$

$$C(unlink\ of\ k\ components) = \begin{cases} 1\ if\ k = 1 \\ 0\ if\ k > 1 \end{cases}$$

It is easy to show (see [2]) that this weight system satisfies the 2-term relations of section 2.2. Simply surger the two chords; the 2-term relation then says just that one band can be "slid" over the other, which doesn't change the topology of the diagram. We will show that this weight system is the same as the the determinant of the intersection graph of the diagram. Our proof is essentially the same as that in [2]; we include it for completeness.

Theorem 4 *[2] For any chord diagram D, $C(D) = det(\Gamma(D))$.*

PROOF: Since both of these weight systems satisfy the 2-term relations, it suffices to show that they agree on caravans. Consider a caravan D with m_1 one-humped camels and m_2 two-humped camels, as shown in Figure 3. Then $adj(\Gamma(D)) \cong [0]^{m_1} \oplus \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^{m_2}$. So $det(\Gamma(D)) = 0^{m_1}1^{m_2} = \begin{cases} 1\ if\ m_1 = 0 \\ 0\ otherwise \end{cases}$.

On the other hand, if we surger all the chords of the diagram, we obtain an unlink with m_1+1 components, which means that $C(D) = \begin{cases} 1\ if\ m_1 = 0 \\ 0\ otherwise \end{cases}$. So the two weight systems agree. $\square$

We now turn to the HOMFLYPT polynomial. We will begin by considering a framed version of the HOMFLYPT polynomial - i.e. an invariant of regular isotopy, rather than isotopy. This invariant is the Laurent polynomial $P(l, m) \in \mathbf{Z}[l^{\pm 1}, m^{\pm 1}]$ defined by the following skein relations (see [9]) (L^+ is the result of adding a positive kink to the link L):

$$P(L_+) - P(L_-) = mP(L_0)$$

$$P(L^+) = lP(L)$$

$$P(L \cup O) = \frac{l - l^{-1}}{m}P(L)$$

$$P(O) = 1$$

If we make the substitutions $m = e^{ax/2} - e^{-ax/2}$ and $l = e^{bx/2}$, and expand the resulting power series, we transform the HOMFLYPT polynomial into a power series in x, whose coefficients are finite type invariants (of regular isotopy). These invariants give rise to regular weight systems which we can collect together as the HOMFLYPT regular weight system H. The skein relations above give rise to the following relations for H, by looking at the first terms of the power series (as before, D_v is the result of surgering the chord v in D):

$$H(D) = aH(D_v)$$

$$H(D \cup O) = bH(D)$$
$$H(O) = 1$$

So if D is an unlink of k components, $H(D) = b^{k-1}$. Since the first of these relations is almost the same as for the Conway weight system C, the same argument shows that H satisfies the 2-term relations. We will use this to show that the HOMFLYPT regular weight system is equivalent to the rank of the intersection graph of the diagram. (This result was found independently by Soboleva, for the case $a = 1$ [15].)

Theorem 5 *For any chord diagram D of degree k, $H(D) = a^k b^{k-rank(\Gamma(D))} = (ab)^k R(\Gamma(D))(b^{-1})$*

PROOF: As with Theorem 4, it suffices to show that the weight systems agree on caravans. Let D be the caravan with m_1 one-humped camels and m_2 two-humped camels, as in Figure 3 (so the degree of D is $m_1 + 2m_2$). As before, $adj(\Gamma(D)) \cong [0]^{m_1} \oplus \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^{m_2}$, so the rank is $2m_2$. On the other hand, if we surger all the chords (each time multiplying H by a), the resulting link has $m_1 + 1$ components, so $H(D) = a^k b^{m_1} = a^k b^{k-rank(\Gamma(D))}$. $\square$

Corollary 2 *If D is a chord diagram of degree k, and L_D is the link with c components obtained by surgering all of the chords of D, then $rank(\Gamma(D)) = k - c + 1$.*

We can also consider the unframed HOMFLYPT polynomial $\widehat{P}(l,m)$, defined by $\widehat{P}(L) = l^{-writhe(L)} P(L)$ (see [9]). This invariant is determined by the skein relations:

$$l\widehat{P}(L_+) - l^{-1}\widehat{P}(L_-) = m\widehat{P}(L_0)$$
$$\widehat{P}(L \cup O) = \frac{l - l^{-1}}{m}\widehat{P}(L)$$
$$\widehat{P}(O) = 1$$

After making the same substitutions as before, we again obtain a power series whose coefficients are finite type invariants (this time of isotopy). The collection of the associated weight systems $\widehat{H}$ was described by Meng [12] (here D_v is the result of surgery on the chord v, and $D \backslash v$ is the result of removing the chord v):

$$\widehat{H}(D) = a\widehat{H}(D_v) - b\widehat{H}(D \backslash v)$$
$$\widehat{H}(D \cup O) = b\widehat{H}(D)$$
$$\widehat{H}(O) = 1$$

It is easy to see that this weight system is simply the canonical projection of H, and so we can conclude that:

Theorem 6 *For any chord diagram D of degree k, $\widehat{H}(D) = (ab)^k \widehat{R}(\Gamma(D))(b^{-1})$*

PROOF: Both weight systems are the canonical projections of H. $\square$

Remark: Rather than considering the rank of the adjacency matrix, we could as easily have studied its nullity. If we define $N(G)(x) = x^{nullity(adj(G))}$, and let $\widehat{N}(G)$ be its canonical projection, then Theorems 5 and 6 imply that $H(D) = a^k N(\Gamma(D))(b)$ and $\widehat{H}(D) = a^k \widehat{N}(\Gamma(D))(b)$.

4 Marked Chord Diagrams and the Kauffman weight system

In this section we will look at *marked* chord diagrams, motivated by the Kauffman polynomial. The idea is that, where we replaced a chord with a band in the previous section, a marked chord will be replaced by a *twisted* band, as in Figure 6. The two different surgeries correspond to the two resolutions of a crossing, L_0 and L_∞, in Figure 4.

We will begin by defining marked chord diagrams and graphs, together with a natural map from the space of chord diagrams (graphs) to the space of marked chord diagrams (graphs). We will define an expanded set of 2-term relations on these spaces, and show that a modification of the adjacency matrix is invariant under these relations. We will use this to construct regular graph weight systems, and show that one of these systems is equivalent to the regular weight system associated with the (framed) Kauffman polynomial.
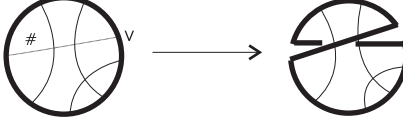
Figure 6: Surgery on a marked chord v

4.1 Marked Chord Diagrams and Graphs

A *marking* of a chord diagram D (respectively, graph G) is simply a partition of the set of chords $J(D)$ (resp. vertices $V(G)$) into two disjoint subsets J_m and J_u (V_m and V_u), where J_m (V_m) is the set of *marked* chords (vertices), and J_u (V_u) is the set of *unmarked* chords (vertices). We will typically denote a marked chord by labeling it with a pound sign (#).

There is a natural map from the vector space of chord diagrams to the vector space of marked chord diagrams, simply taking a diagram to the sum (with signs) of all possible ways of marking it.

Definition 7 *Consider a chord diagram D, and a subset J of the set of chords of D. Let D^J denote the marked chord diagram obtained by marking all the chords in J. Then we define a map M from the vector space of chord diagrams to the vector space of marked chord diagrams by: $M(D) = \sum_{J \subset J(D)} (-1)^{|J|} D^J$.*

We can define a similar map (which we will also denote M) from the vector space of graphs to the space of marked graphs. There is an obvious lifting of the map Γ from a chord diagram to its intersection graph to a map from a marked chord diagram to its marked intersection graph, which we will also denote Γ (simply mark the vertices corresponding to the marked chords). Clearly, for any chord diagram D, $M(\Gamma(D)) = \Gamma(M(D))$.

4.2 Extended 2-term relations

We can extend the 2-term relations from section 2.2 to a set of 2-term relations on the space of marked chord diagrams. Just as the original 2-term relations were motivated by the idea of replacing chords by bands, the extensions are motivated by the idea of replacing marked chords by twisted bands.

The extended 2-term relations are shown in Figure 7. The space of marked chord diagrams modulo these relations can be given the structure of a bialgebra by using the same product and coproduct as in section 2.1. We will denote this bialgebra (and the underlying graded vector space) B^m. The only point which needs to be checked is that the product is well-defined modulo the 2-term relations. This verification is very similar to the corresponding proof for chord diagrams in [1], and is left as an exercise for the reader.

Proposition 1 *If β is a functional on B^m, then $\beta \circ M$ is a regular weight system (a functional on A).*

PROOF: It is easy to check that the image of a 4-term relation under M is a linear combination of 2-term relations, so M is a bialgebra homomorphism from A to B^m. $\square$

We want to find a set of generators for the vector space of marked chord diagrams modulo the 2-term relations. One such spanning set is a generalization of the caravans of the original 2-term relations.

Definition 8 *A marked (n_1, n_2, n_3)-caravan is a marked chord diagram $(\Theta_m)^{n_1} \Theta^{n_2} X^{n_3}$, where Θ_m is the chord diagram consisting of a single marked chord (a marked one-humped-camel), Θ is the diagram consisting of a single unmarked chord (a one-humped-camel), and X is the diagram consisting of two intersecting unmarked chords (a two-humped-camel).*

An example of a marked caravan is shown in Figure 8. We will now show that these caravans span the space of marked chord diagrams, modulo the extended 2-term relations, using an argument similar to that in [2]. In fact, we will show a slightly stronger fact:

Theorem 7 *Any marked chord diagram is equivalent to a marked caravan, modulo the 2-term relations.*
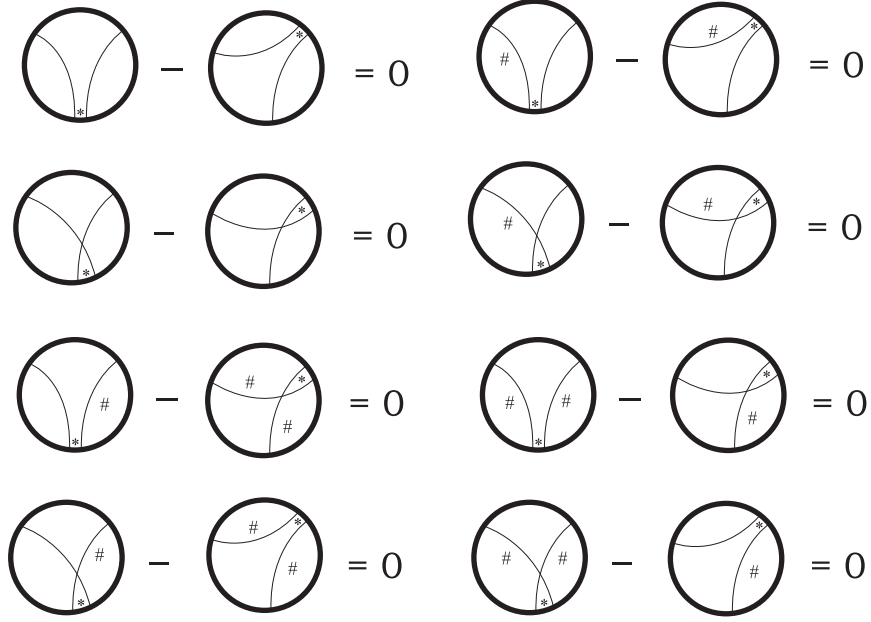
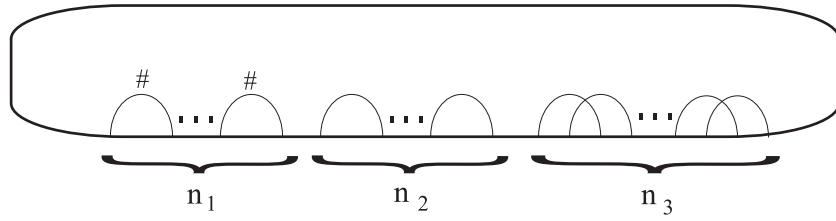Figure 7: 2-term relations on marked chord diagrams

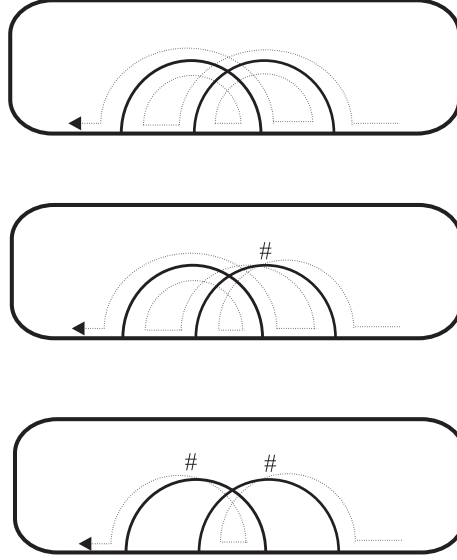Figure 8: A marked (n_1, n_2, n_3)-caravan

Figure 9: Factoring out two-humped-camels. Notice that a chord being slid over the camel will follow the same path whether it is marked or unmarked, and that its marking after being slid over the camel will be the same as before (although it may change during the process).

PROOF: Let D be a marked chord diagram. To begin with, assume that D has two intersecting chords c_1 and c_2 (possibly marked). There are four possibilities: both chords are unmarked, only c_1 is marked, only c_2 is marked, or both chords are marked. In each case, the pair of chords can be slid to the right using the 2-term relations, as in Figure 9, until a (possibly marked) two-humped-camel is factored out. Continuing inductively, we can factor out (possibly marked) two-humped camels until there are no remaining pairs of intersecting chords. Then, among the remaining chords, there will be a "smallest" chord, whose endpoints are not separated by the endpoints of any other chord. This chord, whether marked or unmarked, can be slid to the right as in Figure 10, until a (possibly marked) one-humped-camel is factored out. Continuing inductively, we can reduce the remaining chords to a series of marked and unmarked one-humped-camels. Finally, we can reduce the marked two-humped-camels to pairs of one-humped-camels as in Figure 11. We are left with a product of marked and unmarked one-humped-camels and unmarked two-humped-camels, which is a marked caravan. This completes the proof. $\square$
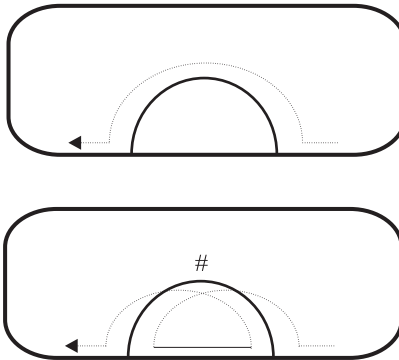

Figure 10: Factoring out one-humped-camels. Notice that a chord being slid over the camel will follow the same path whether it is marked or unmarked, and that its marking after being slid over the camel will be the same as before (although it may change during the process).
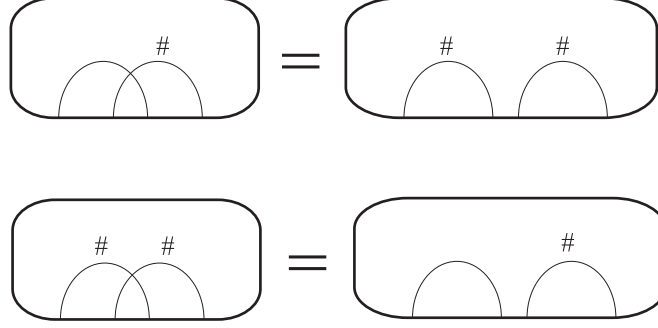
Figure 11: Reducing marked two-humped camels to one-humped camels.

We can define similar 2-term relations for marked graphs. Consider a marked graph G with vertices A, B. Let $(G)_{AB}$ be this graph with both A and B unmarked, $(G)_{A*B}$ be the graph with A marked and B unmarked, $(G)_{AB*}$ be the graph with A unmarked and B marked, and $(G)_{A*B*}$ be the graph with both A and B marked. Then the 2-term relations are $(G'_{AB}, \widetilde{G}_{AB}, \widetilde{G}_{AB}$ are as defined in section 2.4):

$$(G)_{AB} - (\widetilde{G}_{AB})_{AB} = 0$$

$$(G'_{AB})_{AB} - (\widetilde{G}'_{AB})_{AB} = 0$$

$$(G)_{A*B} - (\widetilde{G}_{AB})_{A*B} = 0$$

$$(G'_{AB})_{A*B} - (\widetilde{G}'_{AB})_{A*B} = 0$$

$$(G)_{AB*} - (\widetilde{G}'_{AB})_{A*B*} = 0$$

$$(G'_{AB})_{AB*} - (\widetilde{G}_{AB})_{A*B*} = 0$$

$$(\widetilde{G}_{AB})_{AB*} - (G'_{AB})_{A*B*} = 0$$

$$(\widetilde{G}'_{AB})_{AB*} - (G)_{A*B*} = 0$$

We let E^m denote the vector space of marked graphs modulo these relations. Then E^m can be given the structure of a bialgebra by using the same product and coproduct as in section 2.4.

Proposition 2 *If γ is a functional on E^m, then $\gamma \circ M$ is a regular graph weight system, and $\gamma \circ \Gamma$ is a functional on B^m.*

PROOF: To show the first part of the proposition, we simply need to check that the image of a 4-term relation under M is a linear combination of 2-term relations, so M is a bialgebra homomorphism from F to E^m. The second part of the proposition is immediate. □

The commutative diagram below summarizes the maps between the various bialgebras we have discussed. All of the maps are bialgebra homomorphisms. It is worth noting that the map M is *not* a homomorphism from B to B^m, because the image of a 2-term relation in B may not be a sum of 2-term relations in B^m. The maps p are the natural projections from A and F to B and E, respectively. The maps $\widetilde{p}$ are projections from B^m and E^m to B and E (respectively), defined by sending all diagrams (graphs) with marked chords

(vertices) to 0.

$$
\begin{array}{ccccc}
B^m & \xrightarrow{\quad\Gamma\quad} & & & E^m \\
\big\downarrow\,\widetilde{p} \quad \nwarrow{}^{M} & & & {}^{M}\nearrow & \big\downarrow\,\widetilde{p} \\
& A & \xrightarrow{\ \Gamma\ } & F & \\
\swarrow{}_{p} & & & {}_{p}\searrow & \\
B & \xrightarrow{\quad\Gamma\quad} & & & E
\end{array}
$$

4.3 Marked Adjacency Matrices

Now that we have defined the algebra E^m, and shown that functionals on this algebra give rise to regular graph weight systems and hence (via the deframing map) regular weight systems, we want to construct explicit examples. Once again, we will use the adjacency matrix of a graph. The adjacency matrix of a marked graph is defined as in section 3.1, except that $adj(G)_{ii} = 1$ if v_i is a marked vertex. (We can visualize a marked vertex as having a small loop attached to it, so it is adjacent to itself.)

As in Section 3.1, this matrix can be viewed as a symmetric bilinear form over $\mathbf{Z}_2$, and is well-defined up to isomorphism of forms. As before, we define the *rank* (resp. *det*) of a marked graph as the rank (resp. determinant) of the adjacency matrix of the graph.

Theorem 8 *The isomorphism class of the adjacency matrix of a marked graph satisfies the extended 2-term relations.*

PROOF: Consider a graph G with vertices A and B. We simply need to verify the eight 2-term relations for the adjacency matrix. We can do this by writing down the matrices explicitly, as we did in Theorem 1. As before, we write our matrices with respect to the basis $\{A, B, S_{AB}, S_A, S_B, S_0\}$, and we assume that A and B are connected by an edge in G. Also as before, I and O represent a row or column of 1's and 0's respectively:

$$
adj((G)_{AB}) =
\begin{bmatrix}
0 & 1 & I & I & O & O \\
1 & 0 & I & O & I & O \\
I & I & * & * & * & * \\
I & O & * & * & * & * \\
O & I & * & * & * & * \\
O & O & * & * & * & *
\end{bmatrix}
\cong
\begin{bmatrix}
0 & 1 & O & I & I & O \\
1 & 0 & I & O & I & O \\
O & I & * & * & * & * \\
I & O & * & * & * & * \\
I & I & * & * & * & * \\
O & O & * & * & * & *
\end{bmatrix}
= adj((\widetilde{G}_{AB})_{AB})
$$

$$
adj((G'_{AB})_{AB}) =
\begin{bmatrix}
0 & 0 & I & I & O & O \\
0 & 0 & I & O & I & O \\
I & I & * & * & * & * \\
I & O & * & * & * & * \\
O & I & * & * & * & * \\
O & O & * & * & * & *
\end{bmatrix}
\cong
\begin{bmatrix}
0 & 0 & O & I & I & O \\
0 & 0 & I & O & I & O \\
O & I & * & * & * & * \\
I & O & * & * & * & * \\
I & I & * & * & * & * \\
O & O & * & * & * & *
\end{bmatrix}
= adj((\widetilde{G}'_{AB})_{AB})
$$

$$
adj((G)_{A*B}) =
\begin{bmatrix}
1 & 1 & I & I & O & O \\
1 & 0 & I & O & I & O \\
I & I & * & * & * & * \\
I & O & * & * & * & * \\
O & I & * & * & * & * \\
O & O & * & * & * & *
\end{bmatrix}
\cong
\begin{bmatrix}
1 & 1 & O & I & I & O \\
1 & 0 & I & O & I & O \\
O & I & * & * & * & * \\
I & O & * & * & * & * \\
I & I & * & * & * & * \\
O & O & * & * & * & *
\end{bmatrix}
= adj((\widetilde{G}_{AB})_{A*B})
$$

$$adj((G'_{AB})_{A*B}) = \begin{bmatrix} 1 & 0 & I & I & O & O \\ 0 & 0 & I & O & I & O \\ I & I & * & * & * & * \\ I & O & * & * & * & * \\ O & I & * & * & * & * \\ O & O & * & * & * & * \end{bmatrix} \cong \begin{bmatrix} 1 & 0 & O & I & I & O \\ 0 & 0 & I & O & I & O \\ O & I & * & * & * & * \\ I & O & * & * & * & * \\ I & I & * & * & * & * \\ O & O & * & * & * & * \end{bmatrix} = adj((\widetilde{G}'_{AB})_{A*B})$$

$$adj((G)_{AB*}) = \begin{bmatrix} 0 & 1 & I & I & O & O \\ 1 & 1 & I & O & I & O \\ I & I & * & * & * & * \\ I & O & * & * & * & * \\ O & I & * & * & * & * \\ O & O & * & * & * & * \end{bmatrix} \cong \begin{bmatrix} 1 & 0 & O & I & I & O \\ 0 & 1 & I & O & I & O \\ O & I & * & * & * & * \\ I & O & * & * & * & * \\ I & I & * & * & * & * \\ O & O & * & * & * & * \end{bmatrix} = adj((\widetilde{G}'_{AB})_{A*B*})$$

$$adj((G'_{AB})_{AB*}) = \begin{bmatrix} 0 & 0 & I & I & O & O \\ 0 & 1 & I & O & I & O \\ I & I & * & * & * & * \\ I & O & * & * & * & * \\ O & I & * & * & * & * \\ O & O & * & * & * & * \end{bmatrix} \cong \begin{bmatrix} 1 & 1 & O & I & I & O \\ 1 & 1 & I & O & I & O \\ O & I & * & * & * & * \\ I & O & * & * & * & * \\ I & I & * & * & * & * \\ O & O & * & * & * & * \end{bmatrix} = adj((\widetilde{G}_{AB})_{A*B*})$$

$$adj((\widetilde{G}'_{AB})_{AB*}) = \begin{bmatrix} 0 & 0 & O & I & I & O \\ 0 & 1 & I & O & I & O \\ O & I & * & * & * & * \\ I & O & * & * & * & * \\ I & I & * & * & * & * \\ O & O & * & * & * & * \end{bmatrix} \cong \begin{bmatrix} 1 & 0 & I & I & O & O \\ 0 & 1 & I & O & I & O \\ I & I & * & * & * & * \\ I & O & * & * & * & * \\ O & I & * & * & * & * \\ O & O & * & * & * & * \end{bmatrix} = adj((G)_{A*B*})$$

$$adj((\widetilde{G}_{AB})_{AB*}) = \begin{bmatrix} 0 & 1 & O & I & I & O \\ 1 & 1 & I & O & I & O \\ O & I & * & * & * & * \\ I & O & * & * & * & * \\ I & I & * & * & * & * \\ O & O & * & * & * & * \end{bmatrix} \cong \begin{bmatrix} 1 & 0 & I & I & O & O \\ 0 & 1 & I & O & I & O \\ I & I & * & * & * & * \\ I & O & * & * & * & * \\ O & I & * & * & * & * \\ O & O & * & * & * & * \end{bmatrix} = adj((G'_{AB})_{A*B*})$$

The isomorphisms are just the result of adding the second row (and column) of the matrix on the left to its first row (and column), modulo 2. So the isomorphism classes of the adjacency matrices of the graphs satisfy the extended 2-term relations. □

Corollary 3 *The rank and determinant of a marked graph are functionals on E^m.*

We can combine these functionals with M to obtain regular graph weight systems. In order to construct polynomial-valued weight systems, we will begin with $s(G) = x^{rank(G)}$ and $t(G) = x^{det(G)}$, whose linear extensions are also functionals on E^m:

Theorem 9 *Given an unmarked graph G, and a subset $J \subset V(G)$, define G^J as the result of marking the vertices in J. Define the maps $S(G)$ and $T(G)$ as follows:*

$$S(G)(x) = \sum_{J \subset V(G)} (-1)^{|J|} x^{rank(G^J)}$$

$$T(G)(x) = \sum_{J \subset V(G)} (-1)^{|J|} x^{det(G^J)}$$

Then these maps are regular graph weight systems. Moreover, the map $S(G)$ is multiplicative: $S(G_1 \cdot G_2) = S(G_1)S(G_2)$

PROOF: $S(G) = s(M(G))$ and $T(G) = t(M(G))$ (where s and t are extended linearly), so these are regular graph weight systems by Proposition 2. Since $rank(G_1 \cdot G_2) = rank(G_1) + rank(G_2)$, we can see that $s(G_1 \cdot G_2) = s(G_1)s(G_2)$. It is easy to check that M is also multiplicative. Therefore $S(G)$ is multiplicative. $\square$

Moreover, we can obtain graph weight systems by applying the canonical projection from section 2.4.

Theorem 10 *Given an unmarked graph G with n vertices, a subset $J \subset V(G)$, and a subset $J_m \subset J$, define $G_J^{J_m}$ as the subgraph induced by J, with the vertices in J_m marked. Then we define $\widehat{S}(G)$ and $\widehat{T}(G)$ as follows:*

$$\widehat{S}(G)(x) = \sum_{J \subset V(G)} (x-1)^{n-|J|} S(G_J) = \sum_{J \subset V(G)} \sum_{J_m \subset J} (-1)^{|J_m|}(x-1)^{n-|J|} x^{rank(G_J^{J_m})}$$

$$\widehat{T}(G)(x) = \sum_{J \subset V(G)} T(G_J) = \sum_{J \subset V(G)} \sum_{J_m \subset J} (-1)^{|J_m|} x^{det(G_J^{J_m})}$$

These maps are the canonical projections of $S(G)$ and $T(G)$, and so are graph weight systems.

PROOF: Recall that the deframing map $\phi(G_1 \otimes G_2) = (-\bullet)^{deg(G_1)} \cdot G_2$. It is easy to check that the map $M : F \to E^m$ is multiplicative, i.e. $M(G_1 \cdot G_2) = M(G_1)M(G_2)$. So $M(\phi(G_1 \otimes G_2)) = M(-\bullet)^{deg(G_1)}M(G_2)$. Since $s(G)$ is also multiplicative, we have:

$$
\begin{aligned}
S(\phi(G_1 \otimes G_2)) &= s(M(\phi(G_1 \otimes G_2))) \\
&= s(M(-\bullet))^{deg(G_1)} s(M(G_2)) \\
&= (x-1)^{deg(G_1)} s(M(G_2)) \\
&= (x-1)^{deg(G_1)} S(G_2)
\end{aligned}
$$

From this, it is straightforward to see that the projection of $S(G)$ is $\sum_{J \subset V(G)} (x-1)^{n-|J|} S(G_J)$, as desired.

On the other hand, the determinant of a graph with any isolated unmarked chords is 0. So for T we have (denoting the graph consisting of a single marked vertex by $\bullet\#$):

$$
\begin{aligned}
T(\phi(G_1 \otimes G_2)) &= t(M(\phi(G_1 \otimes G_2))) \\
&= t(M(-\bullet)^{deg(G_1)} M(G_2)) \\
&= t\left((\bullet\# - \bullet)^{deg(G_1)} \sum_{J \subset V(G_2)} (-1)^{|J|} G_2^J \right) \\
&= t\left(\sum_J \sum_{k=0}^{deg(G_1)} \binom{deg(G_1)}{k} (\bullet\#)^k (-\bullet)^{deg(G_1)-k} (-1)^{|J|} G_2^J \right) \\
&= \sum_J \sum_{k=0}^{deg(G_1)} \binom{deg(G_1)}{k} (-1)^{|J|}(-1)^{deg(G_1)-k} t((\bullet\#)^k (\bullet)^{deg(G_1)-k} G_2^J)
\end{aligned}
$$

Since $det((\bullet\#)^k (\bullet)^{deg(G_1)-k} G_x^J) = \begin{cases} det(G_2^J) \ if \ k = deg(G_1) \\ 0 \ otherwise \end{cases}$, we know that $t((\bullet\#)^k (\bullet)^{deg(G_1)-k} G_x^J) = \begin{cases} t(G_2^J) \ if \ k = deg(G_1) \\ 1 \ otherwise \end{cases}$. So our equation reduces to:

$$
\begin{aligned}
T(\phi(G_1 \otimes G_2)) &= \sum_J (-1)^{|J|} \left(t(G_2^J) + \sum_{k=0}^{deg(G_1)-1} \binom{deg(G_1)}{k} (-1)^{deg(G_1)-k} \right) \\
&= \sum_J (-1)^{|J|} \left(t(G_2^J) + (1-1)^{deg(G_1)} - 1 \right) \\
&= \sum_J (-1)^{|J|} t(G_2^J) - \sum_J (-1)^{|J|} \\
&= t(M(G_2)) - 0 = T(G_2)
\end{aligned}
$$

From this, we can conclude that the projection of $T(G)$ is $\widehat{T}(G)(x) = \sum_{J \subset V(G)} T(G_J)$, as desired. □

4.4 The Kauffman weight system

We want to show that $S(\Gamma(D))$ and $\widehat{S}(\Gamma(D))$ are the weight systems associated with the Kauffman polynomial. We will begin by considering a framed version of the Kauffman polynomial $F(y, z)$, defined by the following skein relations (L_+, L_-, L_0 and L_∞ are as shown in Figure 4, and L^+ is the result of adding a positive kink to L):

$$F(L_+) - F(L_-) = z(F(L_0) - F(L_\infty))$$

$$F(L^+) = yF(L)$$

$$F(L \cup O) = \left(\frac{y - y^{-1}}{z} + 1\right) F(L)$$

$$F(O) = 1$$

To derive finite type invariants, we make the substitutions $z = e^{ax/2} - e^{-ax/2}$ and $y = e^{(b-1)x/2}$. If we then expand the polynomial as a power series in x, the coefficients will be finite type invariants. The regular weight system associated with this collection of invariants is defined by the skein relations below. Here D is an unmarked chord diagram, v is a chord in D, D_v is the result of replacing v by an untwisted band, and D^v is the result of replacing v by a band with a half-twist:

$$K(D) = a(K(D_v) - K(D^v))$$

$$K(D \cup O) = bK(D)$$

$$K(O) = 1$$

Note that, if D is an unlink of k components, then $K(D) = b^{k-1}$.

Our first task is to show that this regular weight system factors through the algebra B^m. We define a map $K^m : B^m \to \mathbf{Z}[a, b]$ recursively by the following relations, where D is a marked chord diagram, and v is a chord in D:

$$K^m(D) = \begin{cases} aK^m(D_v) \ \textit{if } v \textit{ is unmarked} \\ aK^m(D^v) \ \textit{if } v \textit{ is marked} \end{cases}$$

$$K^m(D \cup O) = bK^m(D)$$

$$K^m(O) = 1$$

Note that, if D is a diagram with no chords and k components, then $K^m(D) = b^{k-1}$.

Proposition 3 K^m *satisfies the extended 2-term relations.*

PROOF: In each of the 2-term relations of Figure 7, replace each unmarked chord by an untwisted band and each marked chord by a band with a half-twist. It is clear that the relations are simply the result of sliding one band over another, and don't change the topology of the diagram. We need only keep in mind that when a band is slid over a half-twisted band (marked chord), it receives a half-twist itself. We view a band with a full twist as equivalent to an untwisted band, since it does not change the number of components of the diagram, which is all that matters in the base case of the definition of K^m. □

Proposition 4 $K = K^m \circ M$, *so K is the pullback of K^m by M.*

PROOF: Consider a diagram D in A. We will prove the proposition via induction on the number of chords of D. If D has no chords, then $M(D) = D$. Since K and K^m differ only in their first skein relation (which only applies if there are chords), we conclude that $K^m(M(D)) = K^m(D) = K(D)$.

For our inductive step, assume D has a chord v. Note that D_v and D^v each have fewer chords than D, so $K^m(M(D_v)) = K(D_v)$ and $K^m(M(D^v)) = K(D^v)$. If J is a subset of the chords of D, we let D^J denote

the marked chord diagram which results by marking all the chords in J. (However, for the single chord v, we will still let D^v denote the result of replacing v with a half-twisted band.) $M(D)$ is then given by:

$$M(D) = \sum_J (-1)^{|J|} D^J = \sum_{J \ s.t. \ v \notin J} (-1)^{|J|}(D^J - D^{J \cup v})$$

Then:

$$
\begin{aligned}
K^m(M(D)) &= \sum_{J \ s.t. \ v \notin J} (-1)^{|J|}(K^m(D^J) - K^m(D^{J \cup v})) \\
&= \sum_{J \ s.t. \ v \notin J} (-1)^{|J|}(aK^m((D_v)^J) - aK^m((D^v)^J)) \\
&= a(K^m(M(D_v)) - K^m(M(D^v))) \\
&= a(K(D_v) - K(D^v)) \\
&= K(D)
\end{aligned}
$$

So by induction, we conclude that for any diagram D, $K(D) = K^m(M(D))$. $\square$

Theorem 11 *For any $D \in A$ of degree k, $K(D) = (ab)^k S(\Gamma(D))(b^{-1})$.*

PROOF: Since $S(\Gamma(D)) = s(M(\Gamma(D))) = s(\Gamma(M(D)))$, and $K(D) = K^m(M(D))$, it suffices to show that $(ab)^k(s \circ \Gamma(D))(b^{-1}) = K^m(D)$ for any $D \in B^m$. Since both of these maps satisfy the extended 2-term relations, it suffices to show that they agree on marked caravans, by Theorem 7.

Consider a marked (n_1, n_2, n_3)-caravan D, as shown in Figure 8. The degree of this caravan is $k = n_1 + n_2 + 2n_3$. Then $adj(\Gamma(D)) \cong [1]^{n_1} \oplus [0]^{n_2} \oplus \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^{n_3}$. So $rank(\Gamma(D)) = n_1 + 2n_3$, and $(ab)^k s(\Gamma(D))(b^{-1}) = (ab)^k b^{-n_1 - 2n_3} = a^k b^{k - n_1 - 2n_3} = a^k b^{n_2}$.

On the other hand, $K^m(D)$ is computed by replacing all the unmarked chords with untwisted bands and all the marked chords with twisted bands (multiplying by a each time), and then looking at the number of components of the resulting link. This link will have $n_2 + 1$ components, so $K^m(D) = a^k b^{n_2} = (ab)^k s(\Gamma(D))(b^{-1})$, which completes the proof. $\square$

We can also consider the unframed Kauffman polynomial $\widehat{F}(y, z)$, defined by $\widehat{F}(L) = y^{-writhe(L)} F(L)$ (see [9]). This invariant is also determined by the skein relations:

$$y\widehat{F}(L_+) - y^{-1}\widehat{F}(L_-) = m(\widehat{F}(L_0) - \widehat{F}(L_\infty))$$

$$\widehat{F}(L \cup O) = \left(\frac{y - y^{-1}}{z} + 1\right) \widehat{F}(L)$$

$$\widehat{F}(O) = 1$$

After making the same substitutions as before, we again obtain a power series whose coefficients are finite type invariants (this time of isotopy). The collection of the associated weight systems $\widehat{K}$ was described by Meng [12] (here D_v is the result of replacing the chord v by an untwisted band, D^v is the result of replacing the chord v by a half-twisted band, and $D \backslash v$ is the result of removing the chord v):

$$\widehat{K}(D) = a\widehat{K}(D_v) - a\widehat{K}(D^v) - b\widehat{K}(D \backslash v)$$

$$\widehat{K}(D \cup O) = b\widehat{K}(D)$$

$$\widehat{K}(O) = 1$$

It is easy to see that this weight system is simply the canonical projection of K, and so we can conclude that:

Theorem 12 *For any chord diagram D of degree k, $\widehat{K}(D) = (ab)^k \widehat{S}(\Gamma(D))(b^{-1})$*

PROOF: Both weight systems are the canonical projections of K. $\square$

Remark: Rather than considering the rank of the marked adjacency matrix, we could as easily have studied its nullity. If we define $u(G)(x) = x^{nullity(adj(G))}$ and $U(G) = u(M(G))$, and let $\widehat{U}(G)$ be the canonical projection of $U(G)$, then Theorems 11 and 12 imply that $K(D) = a^k U(\Gamma(D))(b)$ and $\widehat{K}(D) = a^k \widehat{U}(\Gamma(D))(b)$.

We now have explicit formulas for computing the Conway, HOMFLYPT and Kauffman weight systems directly from intersection graphs. Hopefully, these interpretations will help shed some light on the geometric meanings of these polynomials.

5   Acknowledgements

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